8) INTERSECTION OF A STRAIGHT LINE AND A CONIC SECTION

When given the equation of a straight line and a conic section, solve simultaneously to get a resulting ***quadratic equation***.

- To find the points of intersection: solve the quadratic equation for x.
- To determine if a line *does* intersect a conic section, do the discriminant test.

The discriminant test:
Your resulting quadratic equation should be in the form $ax^2 + bx + c = 0$.
Find the value of the discriminant $b^2 - 4ac$.

- If $b^2 - 4ac > 0$, (the quadratic has 2 real solns) i.e the line and the conic section intersect at 2 points
- If $b^2 - 4ac = 0$, (the quadratic has one real soln) i.e. the line is a tangent to the conic.
- If $b^2 - 4ac < 0$, (the quadratic has no real soln) i.e. the line does not intersect the conic.

Examples:
1) Find the points of intersection of the line $y = x - 1$ and the hyperbola $x^2 - 4y^2 = 1$.
(note that the hyperbola could be written as $\frac{x^2}{1} - \frac{y^2}{1/4} = 1$)

Substitute $y = x - 1$ into the equation $x^2 - 4y^2 = 1$ and solve for x:
$x^2 - 4(x-1)^2 = 1$
$x^2 - 4(x^2 - 2x + 1) = 1$
$x^2 - 4x^2 + 8x - 4 = 1$
$-3x^2 + 8x - 5 = 0$ solve on Graphics Calculator or use Quadratic Formula
$x = 1, \quad x = 1\frac{2}{3}$

Therefore the points of intersections are $(1, 0)$ and $(1\frac{2}{3}, \frac{2}{3})$.

2) Determine if the line $y = 6x + 12$ intersects the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at 2 points, is a tangent, or does not intersect. Give the coordinate(s) of the point of intersection if applicable.

Substitute $y = 6x + 12$ into $\frac{x^2}{25} + \frac{y^2}{9} = 1$.
$$\frac{x^2}{25} + \frac{(6x+12)^2}{9} = 1$$
$$\frac{x^2}{25} + \frac{36x^2 + 144x + 144}{9} = 1$$
$9x^2 + 25(36x^2 + 144x + 144) = 225$ multiplying both sides by 225
$9x^2 + 900x^2 + 3600x + 3600 = 225$

$909x^2 + 3600x + 3375 = 0$
$101x^2 + 400x + 375 = 0$ dividing by 9

Now do the discriminant test: work out what the value of $b^2 - 4ac$ is for this resulting quadratic:

$a = 101, \quad b = 400, \quad c = 375$
$b^2 - 4ac = 400^2 - 4 \cdot 101 \cdot 375 = 8500$

The value of $b^2 - 4ac$ is positive, so the quadratic has 2 real solutions. Therefore the line intersects the ellipse in 2 places.

Solve the quadratic for x to find the coordinates of the points of intersection:

Using Graphics Calculator or Quadratic Formula:
$x = -1.52, \quad x = -2.44$

The coordinates of the points of intersection are: $(-1.52, 2.88)$ and $(-2.44, -2.64)$.

Worksheet
Delta Ex 38.1 pg 377, Q2, 3
Extension Q4